

$$\begin{aligned}
 J &= \sum_j \left\| z_j - \sum_i \alpha_{ji} e_i \right\|^2 \\
 &= \sum_j \left\| z_j^2 - 2z_j^t \left(\sum_i \alpha_{ji} e_i \right) + \left(\sum_i \alpha_{ji} e_i \right)^2 \right\|^2 \\
 &= \sum_j \left\| z_j \right\|^2 - 2 \sum_j \sum_i \alpha_{ji} z_j^t e_i + \sum_j \sum_i \alpha_{ji}^2
 \end{aligned}$$

$$\frac{\partial J}{\partial \alpha_{me}} = -2 z_m^t e_l + 2 \alpha_{me}$$

$$\Rightarrow \frac{\partial J}{\partial \alpha_{me}} = 0 \quad \text{ssi} \quad \boxed{z_m^t e_l = \alpha_{me}}$$

$$\begin{aligned}
 J &= \sum_j \left\| z_j \right\|^2 - 2 \sum_j \sum_i \underbrace{z_j^t e_i}_{\alpha_{ji}} \underbrace{z_j^t e_i}_{\alpha_{ji}} + \sum_j \sum_i \left(\alpha_{ji} e_i \right)^2 \\
 &= \sum_j \left\| z_j \right\|^2 - \sum_j \sum_i \left(z_j^t e_i \right)^2
 \end{aligned}$$

$$\text{or: } (A^t B)^2 = (B A^t) (A^t B) = B (A A^t) B$$

donc

$$J = \sum_j \left\| z_j \right\|^2 - \sum_j \sum_i e_i^t (z_j z_j^t) e_i$$

$$\begin{aligned}
 J &= \sum_j \|z_j\|^2 - \sum_j \sum_i e_i^t (z_j z_j^t) e_i \\
 &= \sum_j \|z_j\|^2 - \sum_i e_i^t \underbrace{\left(\sum_j z_j z_j^t \right)}_S e_i \\
 &= \sum_j \|z_j\|^2 - \sum_i e_i^t S e_i
 \end{aligned}$$

Minimiser J revient alors à maximiser $\sum_i e_i^t S e_i$ avec la contrainte $e_i^t e_i = 1$.

On utilise alors la méthode des multiplicateurs de Lagrange:

Il faut donc

maximiser $\mathcal{L} = \sum_i e_i^t S e_i - \sum_i \lambda_i (e_i^t e_i - 1)$

Rappel: $\frac{d}{dx} x^t x = 2x$; $\frac{d}{dx} x^t A x = 2Ax$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial e_m} = 2S e_m - \lambda_m 2 e_m$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial e_m} = 0 \text{ssi } \boxed{S e_m = \lambda_m e_m}$$

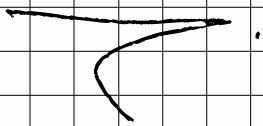
$\Rightarrow \lambda_m$ est une valeur propre de J
et e_m est le vecteur propre associé.

$$\begin{aligned} J &= \sum_j \|z_j\|^2 - \sum_i \lambda_i \overbrace{e_i^t}^t e_i \\ &= \sum_j \|z_j\|^2 - \sum_{i=1}^k \lambda_i \overbrace{e_i^t}^t e_i \\ &= \sum_j \|z_j\|^2 - \sum_{i=1}^k \lambda_i \end{aligned}$$

J sera d'autant plus petite que
les λ_i seront grands.

\Rightarrow Il faut choisir les $\lambda_1, \lambda_2, \dots$

$$\forall \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq \dots \geq \lambda_d.$$



Question: Soit quel la matrice S ?

retour à sa définition:

$$S = \sum_j z_j^t z_j$$

$$\text{on } z_j = x_j - \bar{x}$$

$$\Rightarrow S = \sum_{j=1}^n (x_j - \bar{x})^t (x_j - \bar{x})$$

$$= (n-1) \hat{\Sigma}$$

avec $\hat{\Sigma}$ la matrice de covariance

$$S = (n-1) \sum_{j=1}^n (x_j - \bar{x})^t (x_j - \bar{x})$$

$$= (n-1) \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \dots & \sigma_{1n}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 & \dots & \sigma_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1}^2 & \sigma_{n2}^2 & \dots & \sigma_{nn}^2 \end{pmatrix}$$

λ est valeur propre de S ssi elle est solution de $\det(S - \lambda I) = 0$

$$(n-1)S - \lambda I = \begin{pmatrix} (n-1)\sigma_{11}^2 - \lambda & (n-1)\sigma_{12}^2 & \dots & (n-1)\sigma_{1n}^2 \\ (n-1)\sigma_{21}^2 & (n-1)\sigma_{22}^2 - \lambda & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)\sigma_{n1}^2 & \dots & (n-1)\sigma_{nn}^2 - \lambda & \dots \end{pmatrix}$$